NST Mathematics B

Part IA Mathematical Methods

Comprehensive Tripos Revision Guide

University of Cambridge - Natural Sciences Tripos

Part	Topics	Key Skills
Methods I	ODEs, Linear Algebra, Complex Numbers	Classify & solve ODEs; eigenvalues; complex methods
Methods II	Vector Calculus, Fourier Series	Grad/div/curl; Green's & Stokes' theorems; Fourier expansions
Methods III	PDEs, Probability & Statistics	Separation of variables; wave/heat equations; distributions

0 Contents

0 MATHEMATICAL METHODS I

Ordinary Differential Equations - Linear Algebra - Complex Numbers

1 First-Order ODEs

1.1 Classification Strategy

Always identify the type *before* solving. Work through the hierarchy in order:

Separable → Linear → Exact → Integrating Factor → Bernoulli

1.2 Separable ODEs

Separable ODE

Form: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x)\,g(y)$

$$\frac{\mathrm{d}y}{g(y)} = f(x)\,\mathrm{d}x \quad\Longrightarrow\quad \int \frac{\mathrm{d}y}{g(y)} = \int f(x)\,\mathrm{d}x + C$$

Trap #1

Don't forget the constant of integration C ˇ2014 it appears on *one side only*.

1.3 Linear First-Order ODEs

Integrating Factor Method

Standard form: $\dfrac{\mathrm{d}y}{\mathrm{d}x} + P(x)\,y = Q(x)$

$$\mu(x) = \exp\!\left(\int P(x)\,\mathrm{d}x\right)$$

Multiply through to get $\dfrac{\mathrm{d}}{\mathrm{d}x}\big[\mu(x)\,y\big] = \mu(x)\,Q(x)$, then integrate:

$$\mu(x)\,y = \int \mu(x)\,Q(x)\,\mathrm{d}x + C$$

Key Insight

The left-hand side *always* collapses to $\frac{\mathrm{d}}{\mathrm{d}x}[\mu y]$ ˇ2014 that's the whole point of the integrating factor.

1.4 Exact ODEs

Exact ODE

Form: $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$. **Exact iff** $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Then $\exists\, F(x,y)$ with $\dfrac{\partial F}{\partial x} = M$, $\dfrac{\partial F}{\partial y} = N$, and the solution is $F(x,y) = C$.

$$F(x,y) = \int M\,\mathrm{d}x + g(y), \quad \text{then use } \frac{\partial F}{\partial y} = N \text{ to find } g(y).$$

1.5 Bernoulli Equations

Bernoulli Equation

Form: $\dfrac{\mathrm{d}y}{\mathrm{d}x} + P(x)\,y = Q(x)\,y^n$. Substitute $v = y^{1-n}$:

$$\frac{\mathrm{d}v}{\mathrm{d}x} + (1-n)\,P(x)\,v = (1-n)\,Q(x) \qquad \text{[now linear in } v\text{]}$$

2 Second-Order Linear ODEs

2.1 General Structure

General Solution

$$a\,y'' + b\,y' + c\,y = f(x)$$

$$y = \underbrace{y_{\mathrm{CF}}}_{\text{complementary function}} + \underbrace{y_{\mathrm{PI}}}_{\text{particular integral}}$$

y_{CF} solves the *homogeneous* equation; y_{PI} is any solution of the full equation.

2.2 Complementary Function - Constant Coefficients

Solve the **auxiliary equation** $a\lambda^2 + b\lambda + c = 0$, discriminant $\Delta = b^2 - 4ac$:

Case	**Roots**	**Complementary Function** y_{CF}
$\Delta > 0$	λ_1, λ_2 real, distinct	$A\,e^{\lambda_1 x} + B\,e^{\lambda_2 x}$
$\Delta = 0$	λ repeated	$(A + Bx)\,e^{\lambda x}$
$\Delta < 0$	$\lambda = \alpha \pm i\beta$	$e^{\alpha x}[A\cos(\beta x) + B\sin(\beta x)]$

2.3 Particular Integral - Undetermined Coefficients

$f(x)$	Trial y_{PI}	Caveat
x^n (polynomial)	$a_n x^n + \cdots + a_0$	Include *all* lower powers
e^{kx}	$A\,e^{kx}$	If k is root of AE, try $Ax\,e^{kx}$ or Ax^2e^{kx}
$\cos(\omega x)$ or $\sin(\omega x)$	$A\cos(\omega x) + B\sin(\omega x)$	Use both even if f has only one
$e^{kx}\cos(\omega x)$	$e^{kx}(A\cos\omega x + B\sin\omega x)$	Resonance: multiply by x

Resonance - Most Commonly Missed Trap

If your trial y_{PI} is itself a solution of the homogeneous equation, **multiply by** x (or x^2 if it's a repeated root). Failure to do this is the most common ODE error in Tripos.

2.4 Variation of Parameters

Use when undetermined coefficients fails (e.g. $f(x) = \sec x$, $\tan x$, $\ln x$).

Variation of Parameters

Given y_1, y_2 as the two CF solutions:

$$W = y_1 y_2' - y_2 y_1' \qquad \text{(Wronskian - must be nonzero)}$$

$$y_{\mathrm{PI}} = y_1(x)\int \frac{-y_2\, f(x)}{W}\,\mathrm{d}x \;+\; y_2(x)\int \frac{y_1\, f(x)}{W}\,\mathrm{d}x$$

2.5 Euler-Cauchy Equation

Euler-Cauchy

Form: $x^2y'' + \alpha x\, y' + \beta\, y = 0$

Method 1 - Trial solution: try $y = x^m$. Then m satisfies:

$$m(m-1) + \alpha m + \beta = 0 \qquad \text{(indicial equation)}$$

Method 2 - Substitution: $x = e^t$ converts to a constant-coefficient ODE.

3 Systems of ODEs

A pair of coupled first-order ODEs: $\dot{\mathbf{x}} = A\mathbf{x}$, where $\mathbf{x} = (x, y)^T$.

Solution via Eigenvalues

1. Find eigenvalues λ_1, λ_2 of A: solve $\det(A - \lambda I) = 0$
2. Find eigenvectors $\mathbf{v}_1, \mathbf{v}_2$
3. General solution: $\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$

If $\lambda = \alpha \pm i\beta$, take real and imaginary parts of $e^{\lambda t}\mathbf{v}$.

Defective Matrices (Repeated Eigenvalue, One Eigenvector)

If the matrix is *not* diagonalisable, find a **generalised eigenvector** $\mathbf{v}_2$ satisfying $(A-\lambda I)\mathbf{v}_2 = \mathbf{v}_1$. The solution becomes:

$$\mathbf{x}(t) = e^{\lambda t}\left[C_1\mathbf{v}_1 + C_2(t\,\mathbf{v}_1 + \mathbf{v}_2)\right]$$

4 Linear Algebra

4.1 Matrices and Determinants

Determinants and Inverses

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc, \qquad A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Key properties: $\det(AB) = \det A \det B$; $\det(A^T) = \det A$; $\det(A^{-1}) = 1/\det A$.
A is invertible $\iff$ $\det A \neq 0$.

4.2 Eigenvalues and Eigenvectors

Eigenvalue Problem

$$A\mathbf{v} = \lambda\mathbf{v} \iff (A - \lambda I)\mathbf{v} = \mathbf{0}$$

Characteristic equation: $\det(A - \lambda I) = 0$

For 2×2: $\lambda^2 - \mathrm{tr}(A)\,\lambda + \det(A) = 0$

Checks: $\mathrm{tr}(A) = \lambda_1 + \lambda_2$; $\det(A) = \lambda_1\lambda_2$ - always verify!

Trap

Zero vectors are **never** eigenvectors. If $(A-\lambda I)\mathbf{v} = \mathbf{0}$ has only $\mathbf{v} = \mathbf{0}$, then λ is *not* an eigenvalue.

4.3 Diagonalisation

Diagonalisation

If A has n linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$:

$$A = PDP^{-1}, \quad P = [\mathbf{v}_1 \mid \cdots \mid \mathbf{v}_n], \quad D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$$

$$A^n = PD^nP^{-1}, \qquad e^{At} = P\,e^{Dt}P^{-1}$$

4.4 Symmetric Matrices

Real symmetric matrices ($A = A^T$) have: (i) all *real* eigenvalues; (ii) eigenvectors for *distinct* eigenvalues are orthogonal; (iii) always diagonalisable by an orthogonal matrix P ($P^T = P^{-1}$, $P^TP = I$).

4.5 Quadratic Forms and Positive Definiteness

$Q = \mathbf{x}^TA\mathbf{x}$ (A symmetric). A is **positive definite** iff:

- All eigenvalues > 0, **or**
- All leading principal minors > 0 (Sylvester's criterion)

4.6 Linear Systems $A\mathbf{x} = \mathbf{b}$

Condition	**Solution Type**
$\mathrm{rank}(A) = \mathrm{rank}([A\|\mathbf{b}]) = n$	Unique solution
$\mathrm{rank}(A) = \mathrm{rank}([A\|\mathbf{b}]) < n$	Infinitely many solutions
$\mathrm{rank}(A) < \mathrm{rank}([A\|\mathbf{b}])$	No solution (inconsistent)

5 Complex Numbers & Functions

5.1 Foundations

Forms of a Complex Number

$$z = x + iy = r\,e^{i\theta} = r(\cos\theta + i\sin\theta)$$

$$r = |z| = \sqrt{x^2+y^2}, \qquad \theta = \arg(z) = \mathrm{atan2}(y,x) \in (-\pi, \pi]$$

$$z^* = x - iy = r\,e^{-i\theta}, \qquad z \cdot z^* = |z|^2$$

5.2 Key Identities

Euler's Formula & de Moivre's Theorem

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta) \qquad \text{[de Moivre]}$$

de Moivre Applications

Use de Moivre to express $\cos(n\theta)$ in terms of $\cos\theta$, or to sum trigonometric series. Expand $(e^{i\theta})^n$ by the binomial theorem and equate real/imaginary parts.

5.3 nth Roots

nth Roots of a Complex Number

$z^n = a = R\,e^{i\varphi}$ has n distinct roots:

$$z_k = R^{1/n}\exp\left(\frac{i(\varphi + 2\pi k)}{n}\right), \qquad k = 0, 1, \ldots, n-1$$

nth roots of unity: $\omega_k = e^{2\pi i k/n}$

5.4 Complex Method for Particular Integrals

Complex Shortcut for Trig Forcing

If $y'' + \omega_0^2\,y = \cos(\omega t)$, seek PI as $\mathrm{Re}[z]$ where $z'' + \omega_0^2 z = e^{i\omega t}$.
Trial $z = A\,e^{i\omega t}$:

$$A = \frac{1}{\omega_0^2 - \omega^2} \qquad (\omega \neq \omega_0)$$

Resonance $(\omega = \omega_0)$: $z = \dfrac{t}{2i\omega_0}\,e^{i\omega t}$, so PI $\propto t\sin(\omega t)$ - it grows!

5.5 Complex Logarithm

Complex Logarithm (Multivalued)

$$\log(z) = \ln|z| + i(\theta + 2\pi n), \quad n \in \mathbb{Z}$$

Principal value: $\mathrm{Log}(z) = \ln|z| + i\arg(z)$, $\theta \in (-\pi, \pi]$

$$z^\alpha = \exp(\alpha \log z)$$

Branch Cuts

The complex log is **multivalued**. Always state which branch you are using in exam answers. The principal branch has a cut along the negative real axis.

5 MATHEMATICAL METHODS II

Vector Calculus - Fourier Series & Transforms

6 Vector Algebra

Dot & Cross Products

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = a_1b_1 + a_2b_2 + a_3b_3$$

$$\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}} \qquad \text{(right-hand rule)}$$

$$|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2 \qquad \text{(Lagrange identity)}$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b}) \qquad \text{(BAC-CAB rule)}$$

Triple Products

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \det[\mathbf{a}\,|\,\mathbf{b}\,|\,\mathbf{c}] \qquad \text{(volume of parallelepiped)}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

7 Differential Operators: ∇, div, curl

7.1 Gradient, Divergence, Curl

The Three Operators

Gradient:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right), \qquad \frac{\partial f}{\partial \hat{n}} = \nabla f\cdot\hat{\mathbf{n}} \text{ (directional derivative)}$$

Divergence:

$$\nabla\cdot\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \qquad \text{(source strength)}$$

Curl:

$$\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix} \qquad \text{(local rotation)}$$

7.2 Second-Order Identities

Key Vector Identities (must know)

$$\nabla\cdot(\nabla\times\mathbf{F}) = 0 \qquad \text{(div of curl = 0)}$$

$$\nabla\times(\nabla f) = \mathbf{0} \qquad \text{(curl of gradient = 0)}$$

$$\nabla^2 f = \nabla\cdot(\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \qquad \text{(Laplacian)}$$

$$\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F}$$

$$\nabla\cdot(f\mathbf{F}) = f(\nabla\cdot\mathbf{F}) + \mathbf{F}\cdot\nabla f$$

$$\nabla\times(f\mathbf{F}) = f(\nabla\times\mathbf{F}) - \mathbf{F}\times\nabla f$$

7.3 Curvilinear Coordinates

	Cylindrical (r, φ, z)	**Spherical** (r, θ, φ)
Volume element $\mathrm{d}V$	$r\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}z$	$r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi$
∇f	$\left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \varphi}, \frac{\partial f}{\partial z}\right)$	$\left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\right)$
$\nabla \cdot \mathbf{F}$	$\frac{1}{r}\frac{\partial(rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\varphi}{\partial \varphi} + \frac{\partial F_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial(r^2 F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, F_\theta)}{\partial \theta} + \cdots$
$\nabla^2 f$	$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$	$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \cdots$

8 The Integral Theorems

8.1 Line Integrals and Conservative Fields

Line Integrals

$$W = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C (F_1\,\mathrm{d}x + F_2\,\mathrm{d}y + F_3\,\mathrm{d}z)$$

Conservative field: $\mathbf{F} = \nabla\phi \;\Rightarrow\; \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \phi(B) - \phi(A)$ (path-independent)

$\mathbf{F}$ conservative $\iff \nabla \times \mathbf{F} = \mathbf{0}$ (in a simply-connected region)

8.2 Divergence Theorem (Gauss's Theorem)

Divergence Theorem

$$\oiint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iiint_V (\nabla \cdot \mathbf{F})\,\mathrm{d}V$$

$\hat{\mathbf{n}}$ = *outward* unit normal on closed surface S bounding volume V.

Strategy

If given a closed surface, compute $\iiint \nabla \cdot \mathbf{F}\,\mathrm{d}V$. If $\nabla \cdot \mathbf{F}$ is simple or constant, this is much easier. Also use the reverse: convert a hard volume integral to a surface one.

8.3 Stokes' Theorem

Stokes' Theorem

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S}$$

C = boundary of surface S, oriented by the right-hand rule with respect to $\hat{\mathbf{n}}$.

Orientation

Make sure the orientation of C is consistent with the normal to S - right-hand rule. Getting this wrong flips the sign of your answer.

8.4 Green's Theorem (2D)

Green's Theorem

$$\oint_C (P\,\mathrm{d}x + Q\,\mathrm{d}y) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A$$

Area formula: $A = \dfrac{1}{2}\oint_C (x\,\mathrm{d}y - y\,\mathrm{d}x)$

8.5 Green's Identities

Green's First & Second Identities

First:

$$\int_V \phi\,\nabla^2\psi\,\mathrm{d}V = \oint_S \phi\,(\nabla\psi\cdot\hat{\mathbf{n}})\,\mathrm{d}S - \int_V \nabla\phi\cdot\nabla\psi\,\mathrm{d}V$$

Second:

$$\int_V (\phi\,\nabla^2\psi - \psi\,\nabla^2\phi)\,\mathrm{d}V = \oint_S (\phi\,\nabla\psi - \psi\,\nabla\phi)\cdot\hat{\mathbf{n}}\,\mathrm{d}S$$

9 Fourier Series

9.1 Definition and Coefficient Formulae

Fourier Series (period $2L$)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \qquad n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \qquad n = 1, 2, \ldots$$

Convention Trap

The constant term is $\boldsymbol{a_0/2}$, *not* a_0. The formula gives $a_0 = \frac{1}{L}\int f\,\mathrm{d}x = 2\times\text{mean}$. Examiners check this.

9.2 Even/Odd Symmetry

- $f(x)$ **even**: $b_n = 0, \quad a_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x$
- $f(x)$ **odd**: $a_n = 0, \quad b_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x$

Half-Range Expansions

On $[0, L]$, you *choose* to extend f as even (gives cosine series) or odd (gives sine series). Make this choice explicit in your answer - the resulting series differ!

9.3 Convergence - Dirichlet's Theorem

At a jump discontinuity at $x = x_0$:

$$\text{Fourier series converges to } \frac{f(x_0^-) + f(x_0^+)}{2} \qquad \text{(average of left/right limits)}$$

Gibbs phenomenon: $\approx 9\%$ overshoot near jump discontinuities that *never* disappears as $N \to \infty$.

9.4 Parseval's Theorem

Parseval's Identity

$$\frac{1}{L}\int_{-L}^{L}[f(x)]^2\,\mathrm{d}x = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Classic Tripos Trick - Summing Series via Parseval

To sum $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$: find the Fourier series of $f(x) = x$ (or x^2) on $[-\pi, \pi]$ and apply Parseval. This gives $\dfrac{\pi^2}{6}$. For $\displaystyle\sum \frac{1}{n^4}$, use $f(x) = x^2$.

9.5 Complex Fourier Series

Complex Form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{in\pi x/L}, \qquad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-in\pi x/L}\,\mathrm{d}x$$

Relations: $c_0 = a_0/2$; $c_n = (a_n - ib_n)/2$; $c_{-n} = (a_n + ib_n)/2$

9.6 Fourier Transform

Fourier Transform Pair

$$\tilde{F}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\,\mathrm{d}x \qquad \text{(forward)}$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{F}(k)\, e^{ikx}\,\mathrm{d}k \qquad \text{(inverse)}$$

Parseval: $\displaystyle\int_{-\infty}^{\infty}|f|^2\,\mathrm{d}x = \int_{-\infty}^{\infty}|\tilde{F}|^2\,\mathrm{d}k$

Convolution theorem: $\mathcal{F}[f * g] = \sqrt{2\pi}\,\tilde{F}(k)\,\tilde{G}(k)$, where $(f * g)(x) = \displaystyle\int_{-\infty}^{\infty} f(x-t)\, g(t)\,\mathrm{d}t$

9 MATHEMATICAL METHODS III

Partial Differential Equations - Probability & Statistics

10 PDEs - Classification

Second-Order PDE Classification

General form: $Au_{xx} + Bu_{xy} + Cu_{yy} + \cdots = 0$. Discriminant $\Delta = B^2 - 4AC$:

Type	Δ	Example	Behaviour
Elliptic	< 0	$\nabla^2 u = 0$ (Laplace)	Steady-state; smooth; BVP
Parabolic	$= 0$	$u_t = \kappa u_{xx}$ (Heat)	Diffusion; smoothing; IVP + BCs
Hyperbolic	> 0	$u_{tt} = c^2 u_{xx}$ (Wave)	Wave propagation; finite speed

11 Separation of Variables

11.1 General Strategy

Method of Separation of Variables

1. Assume $u(x,t) = X(x)\,T(t)$; substitute into PDE

2. Rearrange to $\dfrac{X''}{X} = \dfrac{\dot{T}}{\kappa T} = -\lambda$ (separation constant)

3. Solve X ODE with BCs $\to$ eigenvalues λ_n and eigenfunctions $X_n(x)$

4. Solve T ODE for each $\lambda_n \to T_n(t)$

5. General solution: $u = \sum_n c_n X_n(x)\,T_n(t)$

6. Apply ICs using **orthogonality** to find c_n

11.2 Heat Equation

Heat Equation on $[0, L]$

$$u_t = \kappa\, u_{xx}, \qquad u(0,t) = u(L,t) = 0, \qquad u(x,0) = f(x)$$

Eigenfunctions: $X_n = \sin\left(\dfrac{n\pi x}{L}\right)$, eigenvalues $\lambda_n = \left(\dfrac{n\pi}{L}\right)^2$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\kappa \frac{n^2\pi^2 t}{L^2}\right)$$

$$b_n = \frac{2}{L}\int_0^L f(x)\,\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x$$

Long-Time Behaviour

At large t, only the $n = 1$ mode survives: $u \to b_1 \sin(\pi x/L)\, e^{-\kappa\pi^2 t/L^2}$. Higher modes decay faster.

11.3 Wave Equation

Wave Equation on $[0, L]$

$$u_{tt} = c^2\, u_{xx}, \quad u(0,t) = u(L,t) = 0, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

$$\omega_n = \frac{cn\pi}{L}, \qquad u = \sum_{n=1}^{\infty} [A_n \cos(\omega_n t) + B_n \sin(\omega_n t)] \sin\left(\frac{n\pi x}{L}\right)$$

$$A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x, \qquad B_n = \frac{2}{L\omega_n}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x$$

Neumann / Mixed BCs

Neumann BCs ($u_x = 0$ at boundaries) give *cosine* eigenfunctions. Mixed BCs give different eigenvalues. **Always re-derive** - never assume the answer is $\sin(n\pi x/L)$.

11.4 Laplace's Equation

Laplace in a Rectangle $[0, a] \times [0, b]$

$\nabla^2 u = 0$ separates into $X'' + \lambda X = 0$, $Y'' - \lambda Y = 0$.

With $u = 0$ on $x = 0, a$: $X_n = \sin\left(\frac{n\pi x}{a}\right)$, $\quad Y_n = A \sinh\left(\frac{n\pi y}{a}\right) + B \cosh\left(\frac{n\pi y}{a}\right)$

Laplace in a Disk (Polar Coordinates)

$\nabla^2 u = 0$ in $r < a$. Separation in (r, θ):

$$u = a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

Boundedness at $r = 0$ requires r^n terms only (not r^{-n}).

11.5 Sturm-Liouville Theory

The eigenvalue problem from separation: $[p(x)y']' + [q(x) + \lambda w(x)]y = 0$ with BCs.

Sturm-Liouville Key Results

- Eigenvalues λ_n are all **real**
- Eigenfunctions y_n are **orthogonal** with weight $w(x)$: $\displaystyle\int y_m y_n w \,\mathrm{d}x = 0$ for $m \neq n$
- Eigenfunctions are **complete**: any reasonable $f = \sum c_n y_n$

$$c_n = \frac{\displaystyle\int f(x)\, y_n(x)\, w(x)\, \mathrm{d}x}{\displaystyle\int [y_n(x)]^2\, w(x)\, \mathrm{d}x}$$

12 D'Alembert's Solution & Method of Characteristics

D'Alembert's Solution (Infinite Domain)

$u_{tt} = c^2 u_{xx}$ on $-\infty < x < \infty$, with $u(x,0) = f(x)$, $u_t(x,0) = g(x)$:

$$\boxed{u(x,t) = \frac{1}{2}\big[f(x-ct) + f(x+ct)\big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s}$$

Forward wave: $F(x-ct)$; backward wave: $G(x+ct)$.
Characteristics: lines $x \pm ct = \text{const}$.

When to Use D'Alembert

D'Alembert is faster than separation of variables for *infinite* or *semi-infinite* domains with initial conditions only. Use separation of variables for bounded domains with BCs.

13 Probability - Foundations

13.1 Axioms and Key Rules

Probability Axioms & Rules

$$P(A) \geq 0, \quad P(\Omega) = 1, \quad P(A \cup B) = P(A) + P(B) \text{ if } A \cap B = \emptyset$$

$$P(A') = 1 - P(A), \qquad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \qquad A, B \text{ independent} \iff P(A \cap B) = P(A)P(B)$$

$$\textbf{Bayes:} \quad P(B_i|A) = \frac{P(A|B_i)\,P(B_i)}{\sum_j P(A|B_j)\,P(B_j)}$$

13.2 Expectation and Variance

Expectation, Variance, Moments

Discrete: $E[X] = \sum x\,P(X=x)$; **Continuous:** $E[X] = \int x\,f_X(x)\,\mathrm{d}x$

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2$$

$$E[aX+b] = aE[X] + b, \qquad \mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)$$

$$X, Y \text{ independent:} \quad E[XY] = E[X]E[Y], \quad \mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$$

13.3 Key Discrete Distributions

Distribution	PMF $P(X=k)$	$E[X]$	$\mathrm{Var}(X)$	Notes
Bernoulli(p)	$p^k(1-p)^{1-k}$, $k \in \{0,1\}$	p	$p(1-p)$	Single trial
Binomial(n,p)	$\binom{n}{k}p^k(1-p)^{n-k}$	np	$np(1-p)$	n indep. Bernoulli
Poisson(λ)	$e^{-\lambda}\lambda^k/k!$	λ	λ	Limit of Bin, $np = \lambda$
Geometric(p)	$(1-p)^{k-1}p$, $k \geq 1$	$1/p$	$(1-p)/p^2$	Trials to first success
NegBin(r,p)	$\binom{k-1}{r-1}p^r(1-p)^{k-r}$	r/p	$r(1-p)/p^2$	r successes

13.4 Key Continuous Distributions

Distribution	PDF $f_X(x)$	$E[X]$	$\mathrm{Var}(X)$	Notes
Uniform(a,b)	$\frac{1}{b-a}$, $a \leq x \leq b$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Exp(λ)	$\lambda e^{-\lambda x}$, $x \geq 0$	$1/\lambda$	$1/\lambda^2$	Memoryless
Normal(μ,σ^2)	$\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}$	μ	σ^2	Sum of normals is normal
Gamma(α,β)	$\frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha\Gamma(\alpha)}$	$\alpha\beta$	$\alpha\beta^2$	Sum of α Exp r.v.s
$\chi^2(k)$	Gamma($k/2, 2$)	k	$2k$	Sum of k squared normals

13.5 Generating Functions

Moment Generating Function (MGF) & PGF

$$M_X(t) = E[e^{tX}], \qquad M_X^{(n)}(0) = E[X^n]$$

X, Y independent: $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$

PGF (discrete): $G_X(z) = E[z^X] = \sum P(X=k)\, z^k$

$$E[X] = G'(1), \qquad \mathrm{Var}(X) = G''(1) + G'(1) - [G'(1)]^2$$

14 Statistics - Estimation & Hypothesis Testing

14.1 Point Estimation

Sample Statistics

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \text{(sample mean)}$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 \qquad \text{(unbiased sample variance)}$$

$$E[\bar{X}] = \mu, \qquad \mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n}, \qquad E[S^2] = \sigma^2$$

Trap #9

Use $n-1$ (not n) in the denominator of S^2 for the *unbiased* estimator. Using n gives the biased MLE.

14.2 Maximum Likelihood Estimation

MLE Method

$$L(\theta) = \prod_{i=1}^{n} f(x_i|\theta), \qquad \ell(\theta) = \sum_{i=1}^{n} \ln f(x_i|\theta) \quad \text{(log-likelihood)}$$

$$\hat{\theta}_{\mathrm{MLE}} = \arg\max_{\theta} \ell(\theta) \quad \Rightarrow \quad \frac{\mathrm{d}\ell}{\mathrm{d}\theta} = 0$$

Always use the log-likelihood

Maximising $\ell(\theta) = \ln L(\theta)$ gives the same answer as maximising $L(\theta)$, but is far simpler algebraically (products become sums).

14.3 Confidence Intervals

CIs for the Normal Mean

σ **known:** $Z = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$

$$95\%\ \mathrm{CI}: \quad \bar{X} \pm 1.96\,\frac{\sigma}{\sqrt{n}}, \qquad 99\%\ \mathrm{CI}: \quad \bar{X} \pm 2.576\,\frac{\sigma}{\sqrt{n}}$$

σ **unknown:** $T = \dfrac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$

$$95\%\ \mathrm{CI}: \quad \bar{X} \pm t_{n-1,\,0.025} \cdot \frac{S}{\sqrt{n}}$$

14.4 Hypothesis Testing

Test	**Statistic**	**Distribution under H_0**	**Use for**
z-test	$Z = \dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$	$N(0,1)$	Normal, known σ
t-test (1 sample)	$T = \dfrac{\bar{X} - \mu_0}{S/\sqrt{n}}$	t_{n-1}	Normal, unknown σ
2-sample t-test	$T = \dfrac{\bar{X}_1 - \bar{X}_2 - \delta}{SE}$	$t_{n_1+n_2-2}$	Compare two means
χ^2 goodness-of-fit	$X^2 = \sum \dfrac{(O_i - E_i)^2}{E_i}$	χ^2_{k-1}	Categorical data fit
F-test	$F = S_1^2/S_2^2$	F_{n_1-1,n_2-1}	Compare variances

Type I / II Errors

- **Type I error** (false positive): reject H_0 when true - probability $= \alpha$
- **Type II error** (false negative): fail to reject H_0 when false - probability $= \beta$
- **Power** $= 1 - \beta$

χ^2 Test Trap

Always check that *expected* frequency $E_i \geq 5$ in each cell. If not, merge cells before applying the test.

14.5 Simple Linear Regression

Linear Regression $y = \alpha + \beta x$

$$\hat{\beta} = \frac{S_{xy}}{S_{xx}}, \qquad \hat{\alpha} = \bar{y} - \hat{\beta}\,\bar{x}$$

where $S_{xy} = \sum(x_i - \bar{x})(y_i - \bar{y})$, $S_{xx} = \sum(x_i - \bar{x})^2$, $S_{yy} = \sum(y_i - \bar{y})^2$

$$R^2 = \frac{S_{xy}^2}{S_{xx} \cdot S_{yy}} \qquad \text{(coefficient of determination, } 0 \leq R^2 \leq 1\text{)}$$

Residuals: $e_i = y_i - (\hat{\alpha} + \hat{\beta}x_i)$; $\quad s^2 = \sum e_i^2/(n-2)$

14 TRIPOS EXAM STRATEGY

14.5 High-Value Topics

Topic	Likely Question Type
Second-order ODEs with resonance	Full solve + PI via complex method
Eigenvalue problems	Find all eigenvalues/vectors; diagonalise; e^{At}
Divergence Theorem	Convert flux integral to volume integral
Stokes' Theorem	Evaluate line integral via surface integral
Fourier Series + Parseval	Find series; use Parseval to sum $\sum 1/n^2$
Heat/wave equation	Full solution with BCs and ICs
Normal distribution & t-tests	CI construction; hypothesis testing
MLE	Find estimator; show unbiasedness

14.5 General Exam Strategy

Exam Technique

- **Read all questions first.** Identify which parts you can score fully on.
- **Show all working** - method marks are generously awarded even with arithmetic errors.
- **State theorems before applying them** (e.g. "By the Divergence Theorem…").
- **Dimensionally check** answers where possible.
- If integration is hard, check: symmetry, substitution, IBP, partial fractions.

14.5 Top 10 Traps - Commit These to Memory

The 10 Most Common Tripos Errors

1. **ODE**: Forgetting the factor of x for a resonant PI (or x^2 for double root)
2. **Eigenvalues**: Not verifying eigenvectors are linearly independent
3. **Complex log**: Forgetting multivaluedness / branch cuts
4. **Surface integrals**: Wrong sign on normal (outward vs. inward)
5. **Divergence Theorem**: Trying to apply it to an *open* surface
6. **Fourier Series**: Integrating over the wrong range
7. **Parseval**: Confusing $a_0/2$ vs a_0 convention - know your formula sheet
8. **Separation of variables**: Applying BCs to the wrong ODE
9. **Statistics**: Using n instead of $n-1$ in sample variance
10. χ^2 **test**: Forgetting to check expected frequency ≥ 5 in each cell